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New bounds on local strain fields inside random heterogeneous materials ☆

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ABSTRACT

A methodology is presented for bounding the higher L^p norms, $2 \leqslant p \leqslant \infty$, of the local strain inside random media. We present optimal lower bounds that are given in terms of the applied loading and volume fractions for random two phase composites. These bounds provide a means to measure load transfer across length scales relating the excursions of the local fields to applied loads. These results deliver tight upper bounds on the macroscopic strength domains for statistically defined heterogeneous media.

1. Introduction

Failure initiation in heterogeneous media is a multiscale phenomenon. The applied load can be greatly amplified by the local microstructure and can result in local stress and strain concentrations, see for example Kelly and Macmillan (1986). The presence of large local stress and strain often precedes the appearance of nonlinear phenomena such as fracture and yielding. Thus it is crucial to quantify load transfer between length scales when considering failure initiation inside multiscale heterogeneous materials. In this paper we present a new method for quantifying load transfer between length scales when the microstructure is known only in a statistical sense. New tools are provided for teasing out relationships that connect the local strain field to applied macroscopic loads. These relationships provide explicit criteria on the applied loads that are necessary for failure initiation inside statistically defined heterogeneous media. Quantities sensitive to local field behavior include the higher L^p norms of the local strain. These strain measures are used to describe phenomena related to failure initiation inside polymers (Gosse and Christensen, 2001). This paper examines the local strain fields inside statistically homogeneous two phase random elastic media. We address the case when only the volume fractions of the two materials are known. We develop a new methodology for bounding the L^p norms of the local strain inside two phase heterogeneous random media. The method developed in this work is analogous to the method presented in Alali and Lipton (2009) for bounding L^p norms of the local stress inside two phase composites. We use the methods developed here to deliver new explicit lower bounds on the L^p norms of the local strain which are given in terms of the applied loads, volume fractions, and elastic constants of the two materials. Several new lower bounds are presented for a ladder of progressively more complicated macroscopic load cases and are valid for the full range $2 \leqslant p \leqslant \infty$. These bounds are shown to be optimal and provide a means to measure load transfer across length scales relating the variations of the local strain to the applied macroscopic loading. Here we have focused on lower bounds since volume constraints alone do not exclude the existence of microstructures with rough interfaces for which the L^p norms of local fields are divergent see Milton (1986), Faraco (2003) and Leonetti and Nesi (1997). The results presented in this paper provide new quantitative tools

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for the study of failure initiation inside random heterogeneous media. For a given realization of the random medium, the theory of failure initiation posits that failure is initiated when certain rotational invariants of the local elastic strain (or stress) exceed threshold values (Kelly and Macmillan, 1986). Examples include the local hydrostatic strain component ϵ^H and the local deviatoric strain component ϵ^D as well as combinations including $|\epsilon| = \sqrt{(\epsilon^H)^2 + (\epsilon^D)^2}$ see, (Gosse and Christensen, 2001). To fix ideas we introduce the strength domain associated with the norm of the local strain $|\epsilon|$ for two phase statistically homogeneous random elastic media. Here we suppose that only the volume fractions θ_1 and θ_2 of the two elastic materials are known. The strength domain K^{Safe} is defined to be the set of applied constant strains $\bar{\epsilon}$ such that $|\epsilon|$ lies below the failure threshold inside each component material almost surely for every microstructure realization of the random medium with prescribed volume fractions θ_1 and θ_2. An upper bound on the strength domain is defined to be the set $\overline{K}$ of constant strains such that if $\bar{\epsilon}$ lies outside $\overline{K}$ then $|\epsilon|$ has exceeded the threshold over some subset of non zero volume inside one of the component materials for every microstructure composed of materials one and two with prescribed volume fractions θ_1 and θ_2, so

$$K^{Safe} \subset \overline{K}. \tag{1.1}$$

In Section 4 we apply the lower bounds on local fields to obtain explicit, tight upper bounds on the macroscopic strength domain for statistically homogeneous random media. Recent related works provide optimal lower bounds on local stress fields. The work presented in Alali and Lipton (2009) provides new optimal lower bounds on both the local shear stress and the local hydrostatic stress for random media subjected to a series of progressively more general applied macroscopic stresses. These bounds are explicit and given in terms of volume fractions, elastic constants of each phase, and the applied macroscopic stress. The work presented in Chen and Lipton (2010) develops optimal lower bounds on the local hydrostatic stress field inside heterogeneous thermoelastic media undergoing macroscopic thermomechanical loading. These bounds are explicit and given in terms of volume fractions, coefficients of thermal expansion, elastic properties, and the applied macroscopic thermal and mechanical loading. Earlier work provides optimal lower bounds on local fields for random media subjected to applied constant hydrostatic stress and strain and for applied constant electric fields (Lipton, 2004, 2005, 2006). Those efforts deliver optimal lower bounds on the L^p norms for the hydrostatic components of local stress and strain fields as well as the magnitude of the local electric field for all p in the range $2 \leqslant p \leqslant \infty$. Other work examines the stress field around a single simply connected stiff inclusion subjected to a remote constant stress at infinity (Wheeler, 1993) and provides optimal lower bounds for the supremum of the maximum principal stress. The work presented in Grabovsky and Kohn (1995) provides an optimal lower bound on the supremum of the maximum principal stress for two-dimensional periodic composites consisting of a single simply connected stiff inclusion in the period cell. The recent work of (He, 2007) builds on the earlier work of Lipton (2006), Lipton (2005) and develops lower bounds on the L^p norm of the local fields for statistically isotropic two-phase composites. However to date those bounds have been shown to be optimal only for $p = 2$, their optimality for $p > 2$ remains to be seen. Optimal upper and lower bounds on the L^2 norm of local gradient fields are given in Lipton (2001).

The paper is organized as follows. In the next section we present the elastic boundary value problem for heterogeneous media. Section 3 lists optimal lower bounds for a ladder of load cases of increasing generality. The microstructures that support local fields that attain the lower bounds are introduced and discussed in this section. Upper bounds on the strength domains for random media are provided in Section 4. The proofs of the lower bounds on the local strain are derived in Section 5. The attainability of the lower bounds are proved in Section 6.

We conclude by introducing the projections and tensor contractions useful for developing local strain bounds. Generic stress or strain tensor fields are denoted by $\psi(\mathbf{x})$ and $\eta(\mathbf{x})$. Contractions of ψ and η are defined by $\psi : \eta = \psi_{ij}\eta_{ij}$ and $|\psi|^2 = \psi : \psi$, where repeated indices indicate summation. Products of fourth order tensors C and stress or strain tensors ψ are written as $C\psi$ and are given by $[C\psi]_{ij} = C_{ijkl}\psi_{kl}$; and products of stresses or strains η with vectors $\mathbf{v}$ are given by $[\eta\mathbf{v}]_i = \eta_{ij}v_j$. The fourth order identity map on the space of stresses or strains is denoted by $\mathbf{I}$ and $\mathbf{I}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The projection onto the hydrostatic part of $\psi(\mathbf{x})$ is denoted by $\mathbf{\Pi}^H$ and is given explicitly by

$$\mathbf{\Pi}^H_{ijkl} = \frac{1}{d}\delta_{ij}\delta_{kl} \quad \text{and} \quad \mathbf{\Pi}^H\psi(\mathbf{x}) = \frac{\operatorname{tr}\psi(\mathbf{x})}{d}I. \tag{1.2}$$

The projection onto the deviatoric part of $\psi(\mathbf{x})$ is denoted by $\mathbf{\Pi}^D$ and $\mathbf{I} = \mathbf{\Pi}^H + \mathbf{\Pi}^D$ with $\mathbf{\Pi}^D\mathbf{\Pi}^H = \mathbf{\Pi}^H\mathbf{\Pi}^D = 0$. For completeness we introduce the following notation. The rank one matrix formed by taking the outer product of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \otimes \mathbf{b}$ with elements $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_ib_j$. The symmetric part of this matrix is denoted by $\mathbf{a} \odot \mathbf{b}$ with elements $(\mathbf{a} \odot \mathbf{b})_{ij} = (a_ib_j + a_jb_i)/2$.

2. Elastic boundary value problem for heterogeneous media

We present the canonical boundary value problem used to describe local stress and strain fields inside statistically homogeneous random heterogeneous materials (Golden and Papanicolaou, 1983; Jikov et al., 1994; Milton, 2002; Torquato, 2000). Every realization ω of the heterogeneous medium occupies $\mathbf{R}^d$, $d = 2, 3$ and is composed of two elastically isotropic materials with elasticity tensors denoted by C^1 and C^2. The bulk and shear moduli of material one and two are denoted by κ_1 and μ_1, and κ_2 and μ_2 respectively. The isotropic elasticity tensor associated with each component material is given by

$$C^i = 2\mu_i\mathbf{\Pi}^D + d\kappa_i\mathbf{\Pi}^H, \quad \text{for } i = 1, 2, \tag{2.1}$$

where $d = 2$ for planar elastic problems and $d = 3$ for the three dimensional problem. Each realization of the random medium is specified by the indicator functions of phase one and two denoted by $\chi_1(\mathbf{x},\omega)$ and $\chi_2(\mathbf{x},\omega)$. For a given realization $\chi_1(\mathbf{x},\omega)$ takes the value 1 in phase one and zero outside and $\chi_2(\mathbf{x},\omega) = 1 - \chi_1(\mathbf{x},\omega)$. The elastic tensor associated with the two phase medium is denoted by $C(\mathbf{x},\omega)$ and $C(\mathbf{x},\omega) = \chi_1(\mathbf{x},\omega)C^1 + \chi_2(\mathbf{x},\omega)C^2$. Here the index ω belongs to the sample space Ω and the associated probability measure $\mathcal{P}$ is defined over Ω. For the class of statistically homogeneous or strictly spatially stationary and ergodic random media the joint distribution of the sets of indicator functions (for $n = 1, 2, \ldots$)

$$\chi_1(\mathbf{x}_1,\omega), \chi_1(\mathbf{x}_2,\omega), \chi_1(\mathbf{x}_3,\omega), \ldots, \chi_1(\mathbf{x}_n,\omega) \tag{2.2}$$

are invariant under all translations and the ensemble averages of χ_1 coincide with the mean value $\langle\chi_1\rangle$ defined as the limit of volume averages taken over progressively larger volumes (Golden and Papanicolaou, 1983; Jikov et al., 1994; Torquato, 2000). The volume (area) fractions of phase one and two are given by the mean values;

$$\theta_1 = \langle\chi_1\rangle \quad \text{and} \quad \theta_2 = \langle\chi_2\rangle \tag{2.3}$$

and $\theta_1 + \theta_2 = 1$. In what follows we suppress the variable ω when describing solutions associated with a fixed microstructure realization. A constant "macroscopic", strain $\bar{\epsilon}$ is imposed on the heterogeneous material. The local strain is expressed as the sum of a mean zero fluctuation and $\bar{\epsilon}$, i.e., $\epsilon(\mathbf{x}) = \hat{\epsilon}(\mathbf{x}) + \bar{\epsilon}$, with $\langle\hat{\epsilon}\rangle = 0$. The strain fluctuation is given in terms of the displacement field $\hat{\mathbf{u}}$ with $\hat{\epsilon}_{ij}(\mathbf{x}) = (\partial_j\hat{u}_i(\mathbf{x}) + \partial_i\hat{u}_j(\mathbf{x}))/2$. The stress field inside the composite is denoted by $\sigma = \sigma(\mathbf{x})$ and the equation of elastic equilibrium inside each phase is given by

$$\operatorname{div}\sigma = 0. \tag{2.4}$$

The local elastic strain $\epsilon(\mathbf{x})$ is related to the local stress through the constitutive law

$$\sigma(\mathbf{x}) = C(\mathbf{x})\epsilon(\mathbf{x}). \tag{2.5}$$

The traction at an interface with unit normal vector $\mathbf{n}$ pointing into material 2 is denoted by the product $\sigma\mathbf{n}$ and is the vector with components given by $[\sigma\mathbf{n}]_i = \sigma_{ij}n_j$. Perfect contact between the component materials is assumed, thus both the displacement $\hat{\mathbf{u}}$ and traction $\sigma\mathbf{n}$ are continuous across the two phase interface, i.e.,

$$\hat{\mathbf{u}}_{|_1} = \hat{\mathbf{u}}_{|_2}, \tag{2.6}$$
$$\sigma_{|_1}\mathbf{n} = \sigma_{|_2}\mathbf{n}. \tag{2.7}$$

Here the subscripts indicate the side of the interface that the displacement and traction fields are evaluated on. The effective "macroscopic" constitutive law for the random heterogeneous medium is given by the constant effective elasticity tensor C^e (Golden and Papanicolaou, 1983; Jikov et al., 1994; Milton, 2002; Torquato, 2000) relating the average imposed macroscopic strain $\bar{\epsilon}$ to the average stress $\overline{\sigma}$,

$$\overline{\sigma}_{ij} = C^e_{ijkl}\bar{\epsilon}_{kl}. \tag{2.8}$$

In what follows bounds are derived on the moments of the local strain ϵ defined on $\mathbf{R^d}$. Here the moments of a field q are defined to be $\langle|q|^r\rangle^{1/r}$. For future reference we remind the reader that $\lim_{r\to\infty}\langle|q|^r\rangle^{1/r}$ is the same as the $\|q\|_\infty$ norm more commonly defined as the essential supremum of q, see Lieb and Loss (2001).

3. Optimal lower bounds on the local strain inside random composites

In this section we present new optimal lower bounds on the local strain for a ladder of progressively more general sets of applied macroscopic strain. As we progress to more general load cases we will apply additional hypotheses on the shear and bulk moduli of the constituent materials. In this section we provide lower bounds for the following applied macroscopic load cases: (1) lower bounds on the full local strain for applied hydrostatic macroscopic strains, (2) lower bounds on the full local strain inside the material with larger shear modulus for elastic problems with applied macroscopic shear strains, (3) lower bounds on the full local strain for $\mu_1 = \mu_2$, that are seen to be optimal for a special class of applied macroscopic strains, (4) lower bounds on the local deviatoric component of the strain that are optimal for a special class of applied macroscopic strains, and 5) lower bounds on the hydrostatic and deviatoric components of the local strain for the full set of applied macroscopic strains subject to the hypotheses $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$ respectively. In what follows will adopt the notation $\kappa_+ = \max\{\kappa_1, \kappa_2\}, \mu_+ = \max\{\mu_1, \mu_2\}$, $\kappa_- = \min\{\kappa_1, \kappa_2\}$, and $\mu_- = \min\{\mu_1, \mu_2\}$.

For clarity the proofs of the bounds presented in this section are postponed to Section 5. The optimality of these bounds are proved in Section 6.

3.1. Hydrostatic applied strain

In this section we consider applied macroscopic strains that are hydrostatic, i.e., of the form $\bar{\epsilon} = \overline{p}I$ where $\overline{p}$ is a constant and I is the $d \times d$ identity matrix. Here it is assumed that the elastic materials inside the heterogeneous medium are well-ordered i.e., $(\mu_1 - \mu_2)(\kappa_1 - \kappa_2) > 0$ and without loss of generality we will suppose in this section that $\mu_1 > \mu_2$ and $\kappa_1 > \kappa_2$. We present lower bounds that are optimal for all applied hydrostatic strains. We show that the configurations that attain the bounds are given by the Hashin–Shtrikman coated sphere and (cylinder) assemblages (Hashin and Shtrikman, 1962). We now describe the coated sphere assemblage made from a core of material one with a coating of material two and note that the coated cylinder assemblage is constructed similarly. We first fill $\mathbf{R}^3$ with an assemblage of spheres with sizes ranging down to the infinitesimal. Inside each sphere one places a smaller concentric sphere filled with "core" material one and the surrounding coating is filled with material two. The volume fractions of material one and two are taken to be the same for all of the coated spheres. We start by presenting optimal lower bounds on the moments of the local strain inside material one.

Proposition 3.1 (*Optimal lower bounds on the moments of the local strain in material one*). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for an applied hydrostatic macroscopic strain $\bar{\epsilon} = \overline{p}I$ the local strain field inside material one satisfies*

$$\langle\chi_1(\mathbf{x})|\epsilon(\mathbf{x})|^r\langle^{1/r} \geqslant \theta_1^{1/r}\frac{\sqrt{d}(\kappa_2+2\frac{d-1}{d}\mu_2)}{\theta_1\kappa_2+\theta_2\kappa_1+2\frac{d-1}{d}\mu_2}|\overline{p}|,$$
$$\text{for } 2\leqslant r\leqslant\infty. \quad (3.1)$$

Moreover for $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ the lower bound is attained by the local strain inside the coated cylinder (sphere) assemblage with core of material one and coating of material two.

A similar result holds for the local strain field inside material two.

Proposition 3.2 (*Optimal lower bounds on the moments of the local strain in material two*). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for an applied hydrostatic macroscopic strain $\bar{\epsilon} = \overline{p}I$ the local strain field inside material two satisfies*

$$\langle\chi_2(\mathbf{x})|\epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_2^{1/r}\frac{\sqrt{d}(\kappa_1+2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2+\theta_2\kappa_1+2\frac{d-1}{d}\mu_1}|\overline{p}|,$$
$$\text{for } 2\leqslant r\leqslant\infty. \quad (3.2)$$

Moreover for $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ the lower bound is attained by the local strain inside the coated cylinder (sphere) assemblage with core of material two and coating of material one.

The optimal lower bound on the L^∞ norm of the magnitude of the local strain inside a random composite is given by:

Proposition 3.3 (*Optimal lower bounds on the L^∞ norm of the local strain*). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for an applied hydrostatic macroscopic strain $\bar{\epsilon} = \overline{p}I$ the local strain field inside the composite satisfies*

$$\||\epsilon(\mathbf{x})|\|_{L^\infty(Q)} \geqslant \frac{\sqrt{d}(\kappa_1+2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2+\theta_2\kappa_1+2\frac{d-1}{d}\mu_1}\mid\overline{p}\mid. \quad (3.3)$$

Moreover for $d = 2$ the lower bound is attained by the local strain inside the coated cylinder assemblage with core of material two and coating of material one. For $d = 3$ the lower bound is attained by the local strain inside the coated sphere assemblage with core of material two and coating of material one provided that the bulk and shear moduli satisfy the constraint $\kappa_1 \leqslant 3\kappa_2 + 8\mu_1/3$.

Arguments similar to those given in Section 5.1 deliver lower bounds on the local strain field when the two materials are not well ordered, i.e., $\mu_1 > \mu_2$ and $\kappa_1 < \kappa_2$. However explicit calculation shows that the strain fields inside the coated sphere assemblage do not saturate the lower bounds for any combination of core and coating material when the materials are not well-ordered.

3.2. Deviatoric applied strain

In this section the applied macroscopic strains are taken to be purely deviatoric, i.e., $\mathbf{\Pi}^D\bar{\epsilon}^D = \bar{\epsilon}^D$. For two dimensional elastic problems the deviatoric strain tensor can be expressed as the symmetric tensor product of two orthogonal unit vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\bar{\epsilon}^D = \varepsilon(\mathbf{a}\odot\mathbf{b})$, where ε is an arbitrary scalar. In three dimensions this type of strain tensor is referred to as a pure shear strain. For two-dimensional elastic problems we present lower bounds on the local strain that are optimal for all applied deviatoric strains and for three dimensional problems we show that the lower bounds are optimal for any applied pure shear strain. The bounds are attained by simple laminates made by layering material one with material two in the proportions θ_1 and θ_2 respectively. The direction normal to the layers is denoted by $\mathbf{n}$. The optimal choice of layer direction is given by $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$. We present optimal lower bounds on the local strain inside the component material with the larger shear modulus. The volume fraction and indicator functions associated with material having larger shear modulus are denoted by θ_+ and χ_+.

Proposition 3.4 (*Optimal lower bounds on the moments of the local strain inside the phase with larger shear modulus*). *Consider any heterogeneous medium with area (volume) fraction of materials one and two given by θ_1 and θ_2, then for an applied deviatoric macroscopic strain $\bar{\epsilon}^D$ the strain field inside the material with larger shear modulus satisfies*

$$\langle\chi_+|\epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_+^{1/r}\frac{\mu_-}{\theta_1\mu_2+\theta_2\mu_1}|\bar{\epsilon}^D|, \quad \text{for } 2\leqslant r\leqslant\infty. \quad (3.4)$$

Moreover for $d = 2, 3$, when $\bar{\epsilon}^D = \varepsilon(\mathbf{a}\odot\mathbf{b})$ then the lower bound (3.4) is attained by the strain field inside a simple laminate for every r in $2 \leqslant r \leqslant \infty$. Here the layering direction in the optimal laminate is given by $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.

The next result provides a lower bound on the deviatoric component of the local strain inside the material with larger shear modulus.

Proposition 3.5 (*Optimal lower bounds on the moments of the deviatoric component of the local strain inside the material with larger shear modulus*). *Consider any heterogeneous medium with area (volume) fraction of materials one and two given by θ_1 and θ_2, then for an applied deviatoric macroscopic strain $\bar{\epsilon}^D$ the deviatoric component of the local stain field inside the material with larger shear modulus satisfies*

$$\langle\chi_+\mid\mathbf{\Pi}^D\epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_+^{1/r}\frac{\mu_-}{\theta_1\mu_2+\theta_2\mu_1}|\bar{\epsilon}^D|, \quad \text{for } 2\leqslant r\leqslant\infty. \quad (3.5)$$

For $d = 2, 3$, when $\bar{\epsilon}^D = \varepsilon(\mathbf{a}\odot\mathbf{b})$ then the lower bound (3.5) is attained by a simple laminate. The vector normal to the layer interface for the optimal laminate is chosen according to $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$.

3.3. Lower bounds on the local strain that are optimal for a special class of applied macroscopic strain states

In this section we start by considering heterogeneous materials made from two elastic materials sharing the same shear modulus, i.e., $\mu_1 = \mu_2 = \mu$. We present new lower bounds on the full local strain field that hold for every applied macroscopic strain $\bar{\epsilon}$. The lower bounds are shown to be optimal for special subsets $\mathcal{E}_1, \mathcal{E}_2$ of applied strains. The subsets $\mathcal{E}_1, \mathcal{E}_2$ correspond to the set of applied constant strains for which one can construct a confocal ellipsoid assemblage that has constant and purely hydrostatic stress and strain fields inside the core phase (Grabovsky and Kohn, 1995; Milton, 2002). The set $\mathcal{E}_1$ of applied strains is given explicitly by the parametric representation developed in Milton (2002)

$$\bar{\epsilon} = \left(\frac{d\kappa_2 + \frac{(d-1)\mu}{d}}{d^2(\kappa_1 - \kappa_2)}\right) I + \theta_2 M, \tag{3.6}$$

where M ranges over the totality of positive semidefinite $d \times d$ matrices with unit trace. For each $\bar{\epsilon}$ in $\mathcal{E}_1$ one can construct a confocal ellipsoid assemblage with core material one and coating material two such that the local strain inside the core is constant and hydrostatic. Here the axes of the ellipsoids correspond to the principle directions of $\bar{\epsilon}$. The analogous parameterization of the set of applied strains for which the local strain is constant and hydrostatic for suitably constructed confocal ellipsoids with core two is obtained by interchanging subscripts one and two in (3.6). The associated set of macroscopic strains is denoted by $\mathcal{E}_2$. We present optimal lower bounds on the local strain inside material one that hold for all composites with $\mu = \mu_1 = \mu_2$.

Proposition 3.6 (Optimal lower bounds on the local strain inside material one with $\mu_1 = \mu_2$.). *Consider any heterogeneous medium with area (volume) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the strain field inside material one satisfies*

$$\langle \chi_1(\mathbf{x}) |\epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \theta_1^{1/r} \frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} |\mathbf{\Pi}^H \bar{\epsilon}|,$$
$$\text{for } 2 \leqslant r \leqslant \infty. \tag{3.7}$$

Moreover for $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ if $\bar{\epsilon}$ lies in $\mathcal{E}_1$ the lower bound is attained by the local strain inside the confocal-ellipsoid (confocal-ellipse) assemblage.

A similar result holds for the strain fields inside materials two.

Proposition 3.7 (*Optimal lower bounds on the local strain inside material two with $\mu_1 = \mu_2$.). Consider any heterogeneous medium with area (volume) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the strain field inside material two satisfies*

$$\langle \chi_2(\mathbf{x}) |\epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \theta_2^{1/r} \frac{\kappa_1 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} |\mathbf{\Pi}^H \bar{\epsilon}|,$$
$$\text{for } 2 \leqslant r \leqslant \infty. \tag{3.8}$$

Moreover for $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ if $\bar{\epsilon}$ lies in $\mathcal{E}_2$ the lower bound is attained by the local strain inside the confocal-ellipsoid (confocal-ellipse) assemblage.

We conclude this subsection by considering the two trivial lower bounds on the moments of the deviatoric component of the local strain given by $\langle \chi_1(\mathbf{x}) |\mathbf{\Pi}^D \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant 0$ and $\langle \chi_2(\mathbf{x}) |\mathbf{\Pi}^D \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant 0$. In what follows we make no hypothesis on the bulk and shear moduli of the component materials and point out that the trivial bounds are optimal for two subsets of applied macroscopic strains $\bar{\epsilon}$. The subsets are denoted by $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ and these sets correspond to $\mathcal{E}_1$ and $\mathcal{E}_2$ with $\mu = \mu_2$ and $\mu = \mu_1$ respectively. These observations are expressed in the following two propositions.

Proposition 3.8 (*Optimal lower bounds on the deviatoric component of the local strain inside material one). Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ it is evident that the strain field inside material one satisfies*

$$\langle \chi_1(\mathbf{x}) |\mathbf{\Pi}^D \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant 0, \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{3.9}$$

Moreover for $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ if $\bar{\epsilon}$ lies in $\hat{\mathcal{E}}_1$ the lower bound is attained by the local strain inside the confocal-ellipsoid (confocal-ellipse) assemblage with a core of material one.

A similar result holds for strain fields inside material two.

Proposition 3.9 (*Optimal lower bounds on the deviatoric component of the local strain inside material two). Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ it is evident that the strain field inside material two satisfies*

$$\langle \chi_2(\mathbf{x}) |\mathbf{\Pi}^D \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant 0, \text{ for } 2 \leqslant r \leqslant \infty. \tag{3.10}$$

For $d = 2(3)$ and for every r in $2 \leqslant r \leqslant \infty$ if $\bar{\epsilon}$ lies in $\hat{\mathcal{E}}_2$ the lower bound is attained by the local strain inside the confocal-ellipsoid (confocal-ellipse) assemblage with a core of material two.

3.4. Optimal lower bounds for general applied macroscopic strains and $\mu_1 = \mu_2$

In this section we consider two-phase heterogeneous media subject to a general applied macroscopic strain $\bar{\epsilon}$. We suppose that the two materials share the same shear modulus $\mu = \mu_1 = \mu_2$, and we present optimal lower bounds on the hydrostatic part of the local strain. The first result is a lower bound on all moments of the local hydrostatic strain inside each material.

Proposition 3.10 (*Optimal lower bounds on the local hydrostatic strain inside material one with $\mu_1 = \mu_2$ for media subjected to a general applied strain). Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the hydrostatic component of the local strain field inside material one satisfies*

$$\langle \chi_1(\mathbf{x}) |\mathbf{\Pi}^H \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \theta_1^{1/r} \frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} |\mathbf{\Pi}^H \bar{\epsilon}|,$$
$$\text{for } 2 \leqslant r \leqslant \infty. \tag{3.11}$$

Moreover for $d = 2, 3$, the lower bound (3.11) is attained for every r in $2 \leqslant r \leqslant \infty$ by the local hydrostatic strain field inside laminates made from layering the two materials in the prescribed proportions θ_1 and θ_2. Here the layering can be made along any direction $\mathbf{n}$.

A similar result holds for strain fields inside material two.

Proposition 3.11 (Optimal lower bounds on the local hydrostatic strain inside material two with $\mu_1 = \mu_2$ for media subjected to a general applied strain). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the hydrostatic component of the local strain field inside material two satisfies*

$$\langle \chi_2(\mathbf{x}) |\mathbf{\Pi}^H \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \theta_2^{1/r} \frac{\kappa_1 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} |\mathbf{\Pi}^H \bar{\epsilon}|,$$

$$\text{for } 2 \leqslant r \leqslant \infty. \quad (3.12)$$

Moreover for $d = 2, 3$, the lower bound (3.12) is attained for every r in $2 \leqslant r \leqslant \infty$ by the local hydrostatic strain field inside laminates made from layering the two materials in the prescribed proportions θ_1 and θ_2. Here the layering can be made along any direction $\mathbf{n}$.

The next result provides an optimal result on the L^∞ norm of the local strain inside a heterogeneous medium.

Proposition 3.12. (*Optimal lower bounds on the L^∞ norm of the local hydrostatic strain for composites subjected to a general applied strain and $\mu_1 = \mu_2$*). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the hydrostatic component of the local strain field satisfies*

$$\| |\mathbf{\Pi}^H \epsilon(\mathbf{x})| \|_\infty \geqslant \frac{\kappa_+ + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} \mid \mathbf{\Pi}^H \bar{\epsilon} \mid . \quad (3.13)$$

Moreover for $d = 2, 3$, the lower bound (3.13) is attained by the local hydrostatic strain field inside a simply layered material. Here the layering can be made along any direction $\mathbf{n}$.

3.5. Optimal lower bounds for general applied macroscopic strains and $\kappa_1 = \kappa_2$

In this section we consider two-phase heterogeneous media subjected to any applied macroscopic strain $\bar{\epsilon}$. We suppose that the two materials share the same bulk moduli, i.e., $\kappa = \kappa_1 = \kappa_2$. For this case we present optimal lower bounds on the moments of the deviatoric component of the local strain inside the material possessing the largest shear modulus.

Proposition 3.13. (*Optimal lower bounds on the moments of the deviatoric component of the local strain for a general applied macroscopic strain and $\kappa_1 = \kappa_2$*). *Consider any heterogeneous medium with volume (area) fraction of materials one and two given by θ_1 and θ_2, then for any applied macroscopic strain $\bar{\epsilon}$ the deviatoric component of the local strain inside the material with the largest shear stiffness satisfies*

$$\langle \chi_+(\mathbf{x}) |\mathbf{\Pi}^D \epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \theta_+^{1/r} \frac{\mu_-}{\theta_1\mu_2 + \theta_2\mu_1} |\mathbf{\Pi}^D \bar{\epsilon}|,$$

$$\text{for } 2 \leqslant r \leqslant \infty. \quad (3.14)$$

For $d = 2$ let ψ_1, ψ_2 be the orthonormal system of eigenvectors for $\bar{\epsilon}$. Then for every $2 \leqslant r \leqslant \infty$, the lower bound (3.14) is attained by the deviatoric component of the local strain inside a simple laminate with layer normal $\mathbf{n} = \frac{\psi_1 + \psi_2}{\sqrt{2}}$.

Remark. For $d = 3$ a straightforward calculation based upon the explicit solution for the strain field inside a simple laminate given by (6.13)–(6.16) shows that the bound (3.14) is not attained by a simple laminate.

4. Upper bounds on the macroscopic strength domain for random heterogeneous materials

In this section we apply the optimal lower bounds on local strain fields to present new tight upper bounds for strength domains. We begin by considering the case of hydrostatic applied loads of the form $\bar{p}I$. For this case the local strain is of the form $\epsilon(\mathbf{x}) = \bar{p}I + \hat{\epsilon}(\mathbf{x})$ and $\langle \epsilon \rangle = \bar{p}I$. The local stress is related to the local strain through (2.5) and satisfies the equations of elastic equilibrium specified in Section 2. In what follows we present an upper bound on the strength domain associated with norm of the local strain inside the composite. We suppose that failure is initiated inside phase one when $|\epsilon(\mathbf{x})| = F_1$ over some subset of phase one and inside phase two when $|\epsilon(\mathbf{x})| = F_2$ over some subset of phase two. We suppose that only the volume fractions are known, i.e., $\langle \chi_1 \rangle = \theta_1$ and $\langle \chi_2 \rangle = 1 - \theta_1$ and we define the macroscopic strength domain K^{Safe} to be the set of applied strains $\bar{p}I$ for which the local strain field $\epsilon(\mathbf{x})$ satisfies the local constraints

$$\chi_1(\mathbf{x})|\epsilon(\mathbf{x})| < F_1, \quad \chi_2(\mathbf{x})|\epsilon(\mathbf{x})| < F_2. \quad (4.1)$$

We write

$$M_1(\theta_1) = \frac{\sqrt{3}(\kappa_2 + \frac{4}{3}\mu_2)}{\theta_1\kappa_2 + \theta_2\kappa_1 + \frac{4}{3}\mu_2} \quad \text{and}$$

$$M_2(\theta_1) = \frac{\sqrt{3}(\kappa_1 + \frac{4}{3}\mu_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + \frac{4}{3}\mu_1} \quad (4.2)$$

and define the upper bound $\bar{K}$ to be the set of matrices of the form $\bar{p}I$ that satisfy the constraints given by

$$|\bar{p}|M_1(\theta_1) \leqslant F_1 \quad \text{and} \quad |\bar{p}|M_2(\theta_1) \leqslant F_2. \quad (4.3)$$

We now present a tight upper bound on K^{Safe}.

Proposition 4.1 (Upper bound on the macroscopic strength domain for hydrostatic applied loads). *Suppose that $\mu_1 > \mu_2$, $\kappa_1 > \kappa_2, F_1 \geqslant F_2$ and θ_1 is given, then $K^{Safe} \subset \bar{K}$. Moreover $\bar{K}$ is a tight upper bound in that $\bar{p}I \in \bar{K}$ implies that the local strain $|\epsilon(\mathbf{x})|$ does not exceed the failure threshold inside both phases for the coated sphere construction with core material two and coating material one. And $\bar{p}I \notin \bar{K}$ implies that the threshold has been exceeded everywhere inside the core phase of the coated sphere assemblage.*

Proof. Setting $r = \infty$ in (3.1) and (3.2) gives

$$|\overline{p}|M_1(\theta_1) \leqslant \|\chi_1|\epsilon|\|_\infty \quad \text{and} \quad |\overline{p}|M_2(\theta_1) \leqslant \|\chi_2|\epsilon|\|_\infty \tag{4.4}$$

from which the upper bound $K^{Safe} \subset \overline{K}$ follows. Now for the coated sphere assemblage with a core phase of material two an easy computation, using the explicit formula for the local strain field (6.1), shows that $\chi_2(x)|\epsilon(x)| = |\overline{p}|M_2(\theta_1)$. From this observation and the fact that $M_2 > M_1$, the tightness of the upper bound follows. Next consider two-phase heterogeneous media subject to a general imposed macroscopic strain $\bar{\epsilon} = \langle\epsilon\rangle$. The two materials are assumed to share the same shear modulus $\mu = \mu_1 = \mu_2$. The local stress is related to the local strain through (2.5) and satisfies the equations of elastic equilibrium specified in Section 2. We suppose that only the volume fractions are known and we define the macroscopic strength domain K^{Safe} to be the set of applied strains $\bar{\epsilon}$ for which the local strain $\epsilon(\mathbf{x})$ satisfies the local constraints

$$\chi_1(\mathbf{x})|\mathbf{\Pi}^H\epsilon(\mathbf{x})| < F_1, \quad \chi_2(\mathbf{x})|\mathbf{\Pi}^H\epsilon(\mathbf{x})| < F_2. \tag{4.5}$$

We write

$$H_1(\theta_1) = \frac{\kappa_2 + \frac{4}{3}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + \frac{4}{3}\mu} \quad \text{and}$$
$$H_2(\theta_1) = \frac{\kappa_1 + \frac{4}{3}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + \frac{4}{3}\mu} \tag{4.6}$$

and define the upper bound $\overline{K}$ to be the set of matrices $\bar{\epsilon}$ that satisfy the constraints given by

$$\frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_1(\theta_1) \leqslant F_1 \quad \text{and} \quad \frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_2(\theta_1) \leqslant F_2. \tag{4.7}$$

We now present a tight upper bound on K^{Safe}. We note that in this case no assumption is made on either the order of κ_1, κ_2 or the order of F_1, F_2. □

Proposition 4.2 (*Upper bound on the macroscopic strength domain for two-phase media with $\mu_1 = \mu_2$ subjected to a general applied strain*). *Let θ_1 be given. Then $K^{Safe} \subset \overline{K}$. Moreover $\overline{K}$ is a tight upper bound in that $\bar{\epsilon} \in \overline{K}$ implies that the hydrostatic component of the local strain $|\mathbf{\Pi}^H\epsilon(\mathbf{x})|$ does not exceed the failure threshold inside both phases for simple layered materials. And $\bar{\epsilon} \notin \overline{K}$ implies that the threshold has been exceeded everywhere inside at least one of the two phases for simple layered materials. Here the layering can be made along any direction* $\mathbf{n}$.

Proof. Setting $r = \infty$ in (3.11) and (3.12) gives

$$\frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_1(\theta_1) \leqslant \|\chi_1|\mathbf{\Pi}^H\epsilon|\|_\infty \quad \text{and} \quad \frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_2(\theta_1) \leqslant \|\chi_2|\mathbf{\Pi}^H\epsilon|\|_\infty \tag{4.8}$$

from which the upper bound $K^{Safe} \subset \overline{K}$ follows. Now for a simple layered material an easy computation, using the explicit formulas for the strain field inside each phase (6.13)–(6.16), shows that

$$\chi_1|\mathbf{\Pi}^H\epsilon(x)| = \frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_1(\theta_1) \quad \text{and} \quad \chi_2|\mathbf{\Pi}^H\epsilon(x)| = \frac{|\mathrm{tr}\bar{\epsilon}|}{\sqrt{3}}H_2(\theta_1).$$

The tightness of the upper bound follows from these observations. □

5. Lower bounds on the local strain

In this section, we derive the lower bounds on the local strain inside random heterogeneous media listed in Section 3. Their attainability is established in Section 6. The lower bounds are established with the aid of two inequalities that easily follow from Jensen's inequality. Let $\psi(\mathbf{x})$ be a $d \times d$ stress field defined on $\mathbf{R}^d$. Then

$$\langle\chi_i(\mathbf{x})\psi(\mathbf{x}) : \psi(\mathbf{x})\rangle \geqslant \frac{1}{\theta_i}|\langle\chi_i(\mathbf{x})\psi(\mathbf{x})\rangle|^2 \tag{5.1}$$

and

$$\langle\psi(\mathbf{x}) : \psi(\mathbf{x})\rangle \geqslant |\langle\psi(\mathbf{x})\rangle|^2. \tag{5.2}$$

These inequalities are strict in that equality holds in (5.1) only if $\psi(\mathbf{x})$ is constant on the set of points where $\chi_i = 1$ and in (5.2) only if $\psi(\mathbf{x})$ is constant everywhere.

5.1. Proofs of Propositions 3.1–3.3

In this section we suppose that the applied macroscopic strain is hydrostatic, i.e., $\bar{\epsilon} = \overline{p}I$. It is assumed that the elastic materials are well-ordered and we suppose that $\mu_1 > \mu_2$ and $\kappa_1 > \kappa_2$. For this case the lower bounds on the hydrostatic component of the local strain are given by Lipton (2006)

$$\langle\chi_1(\mathbf{x})|\mathbf{\Pi}^H\epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_1^{1/r}\frac{\sqrt{d}(\kappa_2 + 2\frac{d-1}{d}\mu_2)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_2}|\overline{p}|$$
$$\text{for } 2 \leqslant r \leqslant \infty \tag{5.3}$$

and

$$\langle\chi_2(\mathbf{x})|\mathbf{\Pi}^H\epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_2^{1/r}\frac{\sqrt{d}(\kappa_1 + 2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}|\overline{p}|,$$
$$\text{for } 2 \leqslant r \leqslant \infty \tag{5.4}$$

$$\||\mathbf{\Pi}^H\epsilon(\mathbf{x})|\|_\infty \geqslant \frac{\sqrt{d}(\kappa_1 + 2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}\,|\,\overline{p}\,|\,. \tag{5.5}$$

It is pointed out that similar bounds hold for the non-well ordered case (Lipton, 2006). The lower bounds (3.1)–(3.3) follow immediately noting that the norm of the local strain is given by $|\epsilon(\mathbf{x})| = (|\mathbf{\Pi}^H\epsilon(\mathbf{x})|^2 + |\mathbf{\Pi}^D\epsilon(\mathbf{x})|^2)^{1/2}$ so $|\epsilon(\mathbf{x})| \geqslant |\mathbf{\Pi}^H\epsilon(\mathbf{x})|$.

5.2. Proofs of Propositions 3.4 and 3.5

In what follows we make no assumption on the magnitudes of the bulk modulus of each component material. We examine the local strain field inside the material with larger shear modulus and without loss of generality we suppose that $\mu_1 > \mu_2$. We derive new lower bounds on the local strain inside material one that hold for any applied macroscopic deviatoric strain. In subsequent sections these lower bounds are shown to be optimal for applied macroscopic deviatoric strains in two dimensions and for applied macroscopic strains that are pure shear strains in three dimensions. We start by taking $\psi = \Pi^D\sigma$ in Eq. (5.1) to obtain the basic lower bound given by

$$\langle \chi_1 \Pi^D \epsilon(\mathbf{x}) : \Pi^D \epsilon(\mathbf{x})\rangle \geqslant \frac{1}{\theta_1}\left|\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle\right|^2. \tag{5.6}$$

In what follows we obtain a lower bound for the right hand side of (5.6). Applying the definition of the effective elastic tensor gives

$$C^e\bar{\epsilon} = \left\langle \left((C^2 + \chi_1(C^1 - C^2)\right)\epsilon(\mathbf{x})\right\rangle = C^2\bar{\epsilon} + (C^1 - C^2)\langle \chi_1 \epsilon(\mathbf{x})\rangle. \tag{5.7}$$

We apply the deviatoric projection on both sides of equation Eq. (5.7) and solve for $\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle$ to obtain

$$\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle = \frac{1}{2(\mu_1 - \mu_2)}\Pi^D(C^e - C^2)\bar{\epsilon}. \tag{5.8}$$

Up to this point we have assumed that the applied macroscopic strain was given by an arbitrary $d \times d$ matrix. From now on in this subsection we will assume that the applied macroscopic strain is taken to be deviatoric for both two and three dimensional elastic problems, i.e.,

$$\bar{\epsilon} = \bar{\epsilon}^D = \Pi^D\bar{\epsilon}^D \tag{5.9}$$

and one obtains

$$\begin{aligned}\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle &= \frac{1}{2(\mu_1 - \mu_2)}(\Pi^D C^e\bar{\epsilon} - 2\mu_2\Pi^D\bar{\epsilon}) \\ &= \frac{1}{2(\mu_1 - \mu_2)}(\Pi^D C^e \Pi^D\bar{\epsilon} - 2\mu_2\Pi^D\bar{\epsilon}).\end{aligned} \tag{5.10}$$

We apply the Cauchy–Schwarz inequality to find that

$$\left|\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle\right|^2 \geqslant \frac{1}{(2\mu_1 - 2\mu_2)^2}\frac{(C^e\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon} - 2\mu_2\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon})^2}{|\Pi^D\bar{\epsilon}|^2}. \tag{5.11}$$

The effective elasticity tensor satisfies the following well known estimate (Paul, 1960)

$$\begin{aligned}C^e\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon} &\geqslant \langle C^{-1}(\mathbf{x})\rangle^{-1}\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon} \\ &= \frac{2\mu_1\mu_2}{\theta_1\mu_2 + \theta_2\mu_1}|\Pi^D\bar{\epsilon}|^2.\end{aligned} \tag{5.12}$$

Using Eq. (5.12) one obtains

$$C^e\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon} - 2\mu_2\Pi^D\bar{\epsilon} : \Pi^D\bar{\epsilon} \geqslant \frac{\theta_1\mu_2(\mu_1 - \mu_2)}{\theta_1\mu_2 + \theta_2\mu_1}\left|\Pi^D\bar{\epsilon}\right|^2. \tag{5.13}$$

Because $\mu_1 > \mu_2$, and after some simplification, we obtain from Eqs. (5.11) and (5.13) that

$$\left|\langle \chi_1 \Pi^D \epsilon(\mathbf{x})\rangle\right|^2 \geqslant \frac{\theta_1^2\mu_2^2}{(\theta_1\mu_2 + \theta_2\mu_1)^2}\left|\Pi^D\bar{\epsilon}\right|^2 \tag{5.14}$$

and it follows from Eq. (5.6) that

$$\langle \chi_1 |\Pi^D \epsilon(\mathbf{x})|^2\rangle \geqslant \theta_1\frac{\mu_2^2}{(\theta_1\mu_2 + \theta_2\mu_1)^2}\left|\Pi^D\bar{\epsilon}\right|^2. \tag{5.15}$$

For p and q such that $p \geqslant 1$ and $1/p + 1/q = 1$, we apply Hölder's inequality to find that

$$\theta_1^{1/q}\langle \chi_1 |\Pi^D \epsilon(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \langle \chi_1 |\Pi^D \epsilon(\mathbf{x})|^2\rangle \tag{5.16}$$

and hence the inequality

$$\langle \chi_1 |\Pi^D \epsilon(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_1^{1/p}\left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}\right)^2\left|\Pi^D\bar{\epsilon}\right|^2, \tag{5.17}$$

for $1 \leqslant p \leqslant \infty$. The bound (3.5) now follows immediately from (5.17). The bound (3.4) also follows from (5.17) and on noting that

$$\langle \chi_1 \mid \epsilon(\mathbf{x})|^r\rangle^{1/r} \geqslant \langle \chi_1 \mid \Pi^D\epsilon(\mathbf{x})|^r\rangle^{1/r}, \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{5.18}$$

5.3. Proofs of Propositions 3.6, 3.7, 3.10–3.12

In this subsection the applied macroscopic strain is assumed to be any constant $d \times d$ strain tensor, $d = 2, 3$. In what follows we suppose that the two component materials share the same shear modulus, i.e., $\mu = \mu_1 = \mu_2$, and we derive the lower bounds given by (3.7), (3.8), (3.11)–(3.13). In Section 6 the lower bounds on the full local strain are shown to be optimal for special sets $\mathcal{E}_1$ and $\mathcal{E}_2$ and the lower bounds on the hydrostatic component of the local strain is shown to be optimal for all applied macroscopic strains. The dilatational strain inside material one satisfies the following estimate

$$\langle \chi_1 \Pi^H \epsilon(\mathbf{x}) : \Pi^H \epsilon(\mathbf{x})\rangle \geqslant \frac{1}{\theta_1}\left|\langle \chi_1 \Pi^H \epsilon(\mathbf{x})\rangle\right|^2, \tag{5.19}$$

which can be seen by taking $\psi = \Pi^H\epsilon$ in Eq. (5.1). From Eq. (5.7) and since $\mu_1 = \mu_2$, one obtains

$$C^e\bar{\epsilon} = C^2\bar{\epsilon} + 2(\kappa_1 - \kappa_2)\Pi^H\langle \chi_1 \epsilon(\mathbf{x})\rangle. \tag{5.20}$$

For a composite consisting of two isotropic phases of equal shear moduli ($\mu_1 = \mu_2 = \mu$), Hill's relation (Hill, 1963) gives

$$C^e = 2\mu\Pi^D + d\kappa^e\Pi^H, \tag{5.21}$$

where

$$\kappa^e = (\theta_1\kappa_1 + \theta_2\kappa_2) - \frac{\theta_1\theta_2(\kappa_1 - \kappa_2)^2}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}. \tag{5.22}$$

Substitution of (5.21) into (5.20) and solving for $\Pi^H\langle \chi_1 \epsilon(\mathbf{x})\rangle$ gives

$$\Pi^H\langle \chi_1 \epsilon(\mathbf{x})\rangle = \frac{\kappa^e - \kappa_2}{\kappa_1 - \kappa_2}\Pi^H\bar{\epsilon}. \tag{5.23}$$

From estimate (5.19) we recover

$$\langle \chi_1 \Pi^H \epsilon(\mathbf{x}) : \Pi^H \epsilon(\mathbf{x})\rangle \geqslant \frac{1}{\theta_1}\left(\frac{\kappa^e - \kappa_2}{\kappa_1 - \kappa_2}\right)^2\left|\Pi^H\bar{\epsilon}\right|^2 \tag{5.24}$$

and using the formula for κ^e given by (5.22), we express (5.24) as

$$\langle \chi_1 \Pi^H \epsilon(\mathbf{x}) : \Pi^H \epsilon(\mathbf{x})\rangle \geqslant \theta_1\left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}\right)^2\left|\Pi^H\bar{\epsilon}\right|^2. \tag{5.25}$$

An application of Hölder's inequality to (5.25) delivers

$$\langle \chi_1 |\mathbf{\Pi}^H \epsilon(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_1^{1/p} \left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu} \right)^2 |\mathbf{\Pi}^H \bar{\epsilon}|^2, \quad (5.26)$$

for $1 \leqslant p \leqslant \infty$, and the bound Eq. (3.11) follows. Identical arguments give lower bounds on the moments of the hydrostatic strain inside phase two, bound (3.12). The L^∞ bound, Eq. (3.13), follows from the bounds (3.11) and (3.12) by taking $r = \infty$ noting that $\||\mathbf{\Pi}^H \epsilon(\mathbf{x})|\|_\infty \geqslant \|\chi_i |\mathbf{\Pi}^H \epsilon(\mathbf{x})|\|_\infty$ for $i = 1, 2$. The bounds (3.7) and (3.8) follow from the bounds (3.11) and (3.12) and the fact that

$$\langle \chi_i(\mathbf{x}) |\epsilon(\mathbf{x})|^r \rangle^{1/r} \geqslant \langle \chi_i(\mathbf{x}) |\mathbf{\Pi}^H \epsilon(\mathbf{x})|^r \rangle^{1/r}. \quad (5.27)$$

5.4. Proof of Proposition 3.13

In this subsection no constraints are placed on the applied macroscopic strain. The applied macroscopic strain can be any constant $d \times d$ stress tensor, $d = 2, 3$. In what follows we suppose that the two component materials share the same bulk modulus, i.e., $\kappa = \kappa_1 = \kappa_2$ and we derive new lower bounds on the local Von Mises strain inside the material with greater shear stiffness. To fix ideas we suppose that material one has the greater shear stiffness, i.e., $\mu_1 > \mu_2$. We will establish the lower bound Eq. (3.14) with the aid of the following observation whose proof is provided in Alali and Lipton (2009).

Form of effective stiffness tensor for mixtures of two elastically isotropic materials having common bulk modulus.

For $\kappa = \kappa_1 = \kappa_2$, the effective elasticity tensor C^e can be written as

$$C^e = \mathbf{\Pi}^D C^e \mathbf{\Pi}^D + d\kappa \mathbf{\Pi}^H. \quad (5.28)$$

Choosing $\psi = \mathbf{\Pi}^D \epsilon$ in Eq. (5.1) gives

$$\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) : \mathbf{\Pi}^D \epsilon(\mathbf{x}) \rangle \geqslant \frac{1}{\theta_1} |\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) \rangle|^2. \quad (5.29)$$

We notice from Eq. (5.28) that C^e commutes with $\mathbf{\Pi}^D$. Thus Eq. (5.8) becomes

$$\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) \rangle = \frac{1}{2(\mu_1 - \mu_2)} (C^e - 2\mu_2) \mathbf{\Pi}^D \bar{\epsilon} \quad (5.30)$$

and we apply the Cauchy–Schwarz inequality to find that

$$|\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) \rangle|^2 \geqslant \frac{1}{(2\mu_1 - 2\mu_2)^2} \frac{(C^e \mathbf{\Pi}^D \bar{\epsilon} : \mathbf{\Pi}^D \bar{\epsilon} - 2\mu_2 \mathbf{\Pi}^D \bar{\epsilon} : \mathbf{\Pi}^D \bar{\epsilon})^2}{|\mathbf{\Pi}^D \bar{\epsilon}|^2}. \quad (5.31)$$

Application of (5.12) to (5.31) gives

$$C^e \mathbf{\Pi}^D \bar{\epsilon} : \mathbf{\Pi}^D \bar{\epsilon} - 2\mu_2 \mathbf{\Pi}^D \bar{\epsilon} : \mathbf{\Pi}^D \bar{\epsilon} \geqslant \frac{\theta_1 \mu_2 (\mu_1 - \mu_2)}{\theta_1 \mu_2 + \theta_2 \mu_1} |\mathbf{\Pi}^D \bar{\epsilon}|^2. \quad (5.32)$$

We easily see from Eqs. (5.31) and (5.32) that

$$|\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) \rangle|^2 \geqslant \frac{\theta_1^2 \mu_2^2}{(\theta_1 \mu_2 + \theta_2 \mu_1)^2} |\mathbf{\Pi}^D \bar{\epsilon}|^2 \quad (5.33)$$

and it follows from Eq. (5.29) that

$$\langle \chi_1 \mathbf{\Pi}^D \epsilon(\mathbf{x}) : \epsilon(\mathbf{x}) \rangle \geqslant \theta_1 \frac{\mu_2^2}{(\theta_1 \mu_2 + \theta_2 \mu_1)^2} |\mathbf{\Pi}^D \bar{\epsilon}|^2. \quad (5.34)$$

The bound (3.14) follows immediately from Hölder's inequality applied to the left hand side of (5.34).

6. Microstructures that support optimal local fields

It is well known that the coated sphere, coated ellipsoid and laminated microstructures possess optimal effective elastic properties, for reviews of the literature see Milton (2002), Torquato (2000). In the following subsections we show that these microstructures possess optimal local field properties as well.

6.1. The coated sphere construction and optimal lower bounds on local strain fields

In this section, it is shown that the lower bounds presented in Section (3.1) are attained by the stress fields inside the Hashin–Shtrikman (Hashin, 1962; Hashin and Shtrikman, 1962) coated cylinder and sphere assemblages, see Fig. 1. We introduce the normalized L^p norm of a field f over a domain S by $(|S|^{-1} \int_S |f(\mathbf{x})|^p d\mathbf{x})^{1/p}$. One striking feature of the fields inside the coated sphere and cylinder assemblage is that the normalized L^p norm of the local stress or strain taken over a prototypical coated cylinder or sphere is the same as the L^p norm of the whole assemblage. Thus the L^p norms of local fields inside these assemblages are obtained by computing the L^p norm of a prototypical coated sphere or disk. Assume that the applied field $\bar{\epsilon}$ is hydrostatic, $\bar{\epsilon} = \bar{p}I$. The strain field inside a prototypical coated sphere (cylinder) with core of material two and coating of material one in Hashin–Shtrikman assemblage, is given by

$$\epsilon = \begin{cases} \bar{p}A_1 I - \bar{p}A_2 \left(\frac{d\hat{\mathbf{x}} \otimes \hat{\mathbf{x}} - I}{|\mathbf{x}|^d} \right), & a < |\mathbf{x}| \leqslant b, \\ \bar{p}A_3 I, & |\mathbf{x}| \leqslant a \end{cases} \quad (6.1)$$

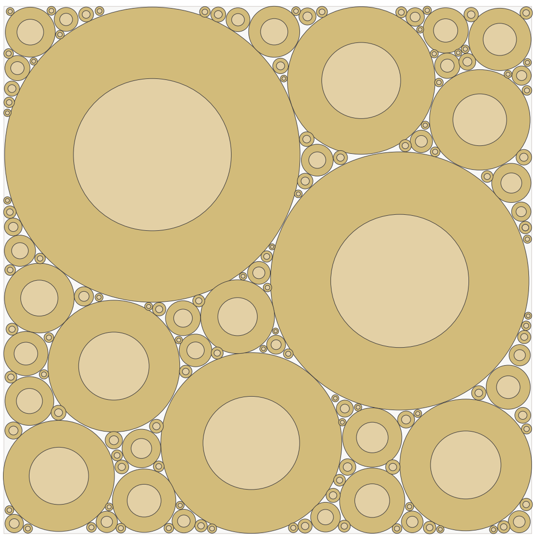

Fig. 1. Hashin–Shtrikman coated cylinder assemblage.

and the constants A_1, A_2, A_3 are given by

$$A_1 = \frac{\kappa_2 + 2\frac{d-1}{d}\mu_1}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}, \tag{6.2}$$

$$A_2 = \frac{-a^d(\kappa_2 - \kappa_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}, \tag{6.3}$$

$$A_3 = \frac{\kappa_1 + 2\frac{d-1}{d}\mu_1}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}. \tag{6.4}$$

We see from Eq. (6.1) that the strain field inside the core material (material two) is hydrostatic, thus

$$\langle \chi_2(\mathbf{x})|\epsilon(\mathbf{x})|^r\rangle^{1/r} = \langle \chi_2(\mathbf{x})|\mathbf{\Pi}^H\epsilon(\mathbf{x})|^r\rangle^{1/r}. \tag{6.5}$$

On the other hand this microstructure attains the lower bound (5.4) see Lipton (2006). Optimality of the lower bound (3.2) follows from these observations. Similar arguments show the lower bound (3.1) is attained by the strain field inside material one of a coated sphere (cylinder) assemblage with core of material one and coating of material two. To show that the strain field inside the coated sphere (cylinder) assemblage with core phase two and coating phase one attains the L^∞ bound (3.3) we use Eqs. (6.1)–(6.4) to compute the maximum strain inside each material. It is found that

$$\|\chi_1|\epsilon|\|_\infty = \frac{\sqrt{(\kappa_2 + 2\frac{d-1}{d}\mu_1)^2 + (d-1)(\kappa_1 - \kappa_2)^2}}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}\sqrt{d}|\overline{p}|, \tag{6.6}$$

$$\|\chi_2|\epsilon|\|_\infty = \frac{\sqrt{d}(\kappa_1 + 2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}|\overline{p}|. \tag{6.7}$$

A straight forward calculation shows that

$$\|\chi_2|\epsilon|\|_\infty^2 - \|\chi_1|\epsilon|\|_\infty^2 = (\kappa_1 - \kappa_2)\left(((2-d)\kappa_1 + d\kappa_2 + 4\left(\frac{d-1}{d}\mu_1\right)\right) \times \frac{d}{(\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1)^2}|\overline{p}|^2. \tag{6.8}$$

It follows from Eq. (6.8) that if $d = 3$ and the elastic materials satisfy $\kappa_1 < 3\kappa_2 + 8\mu_1/3$ or if $d = 2$, then $\|\chi_2|\epsilon|\|_\infty \geqslant \|\chi_1|\epsilon|\|_\infty$ so

$$\||\epsilon|\|_\infty = \|\chi_2|\epsilon|\|_\infty = \frac{\sqrt{d}(\kappa_1 + 2\frac{d-1}{d}\mu_1)}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu_1}|\overline{p}| \tag{6.9}$$

and it is evident that the bound (3.3) is attained by the local fields inside the coated sphere (cylinder) assemblage.

6.2. The strain field inside simple laminates and optimal bounds on local fields

For a laminate made from two isotropic phases the local strain field is piecewise constant under uniform applied strain $\bar{\epsilon}$. Thus

$$\bar{\epsilon} = \langle \chi_1(\mathbf{x})\epsilon(\mathbf{x}) + \chi_2(\mathbf{x})\epsilon(\mathbf{x})\rangle = \theta_1\bar{\epsilon}^1 + \theta_2\bar{\epsilon}^2 \tag{6.10}$$

where $\bar{\epsilon}^i$ is the (constant) field inside the i-th phase. Since the strain field inside each phase satisfies the equation of elastic equilibrium equation (2.4) and from the continuity of the displacement $\mathbf{u}$ and the traction $\sigma\mathbf{n}$ across the two phase interface Eqs. (2.6) and (2.7), it follows that

$$(C^1\bar{\epsilon}^1)\mathbf{n} = (C^2\bar{\epsilon}^2)\mathbf{n}, \tag{6.11}$$

$$\bar{\epsilon}^1 - \bar{\epsilon}^2 = \lambda \odot \mathbf{n}, \tag{6.12}$$

where λ is a vector to be determined and $\mathbf{n}$ is the layering direction of the laminate. Solution of the system of Eqs. (6.10)–(6.12) delivers the local strain field inside each layer. The fields are given by

$$\bar{\epsilon}^1 = \bar{\epsilon} + \theta_2\lambda \odot \mathbf{n}, \tag{6.13}$$

$$\bar{\epsilon}^2 = \bar{\epsilon} - \theta_1\lambda \odot \mathbf{n} \tag{6.14}$$

and

$$\lambda \odot \mathbf{n} = A(\bar{\epsilon}\mathbf{n} \odot \mathbf{n}) - \left(B(\bar{\epsilon}\mathbf{n}\cdot\mathbf{n}) + C\frac{\mathrm{tr}\bar{\epsilon}}{d}\right)\mathbf{n} \odot \mathbf{n}. \tag{6.15}$$

Here

$$A = \frac{2\Delta\mu}{\langle\tilde{\mu}\rangle},$$
$$B = \frac{2\Delta\mu(d\langle\tilde{\kappa}\rangle + (d-2)\langle\tilde{\mu}\rangle)}{\langle\tilde{\mu}\rangle((2d-2)\langle\tilde{\mu}\rangle + d\langle\tilde{\kappa}\rangle)},$$
$$C = \frac{d(2\Delta\mu + d\Delta\kappa)}{(2d-2)\langle\tilde{\mu}\rangle + d\langle\tilde{\kappa}\rangle}, \tag{6.16}$$

where $\langle\tilde{\mu}\rangle = \theta_1\mu_2 + \theta_2\mu_1$, $\langle\tilde{\kappa}\rangle = \theta_1\kappa_2 + \theta_2\kappa_1$, $\langle\mu\rangle = \theta_1\mu_1 + x\theta_2\mu_2$, $\langle\kappa\rangle = \theta_1\kappa_1 + \theta_2\kappa_2$, $\Delta\mu = \mu_1 - \mu_2$, and $\Delta\kappa = \kappa_1 - \kappa_2$. We recall that both deviatoric applied strain in two dimensions as well as pure shear strain in three dimensions can be expressed in the form $\bar{\epsilon} = \varepsilon(\mathbf{a} \odot \mathbf{b})$ with $\mathbf{a}\cdot\mathbf{b} = 0$, $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 1$. On choosing $\mathbf{n} = \mathbf{a}$ or $\mathbf{n} = \mathbf{b}$ in (6.15) one easily finds that

$$\lambda \odot \mathbf{n} = -\frac{\Delta\mu}{\langle\tilde{\mu}\rangle}\bar{\epsilon} \tag{6.17}$$

and it follows from Eq. (6.13) that

$$\bar{\epsilon}^1 = \frac{\mu_2}{\langle\tilde{\mu}\rangle}\bar{\epsilon}. \tag{6.18}$$

From this observation it is evident that the strain field inside this simple laminate attains the bounds (3.4) and (3.5). When both materials share the same shear modulus we find that the local hydrostatic strain fields inside simple laminates have extremal properties. We demonstrate that the lower bounds (3.11)–(3.13) are attained by the hydrostatic strain fields inside any simple laminate. For a simple laminate the strain field inside each material is constant hence both sides of inequality (5.19) are in fact equal

$$\langle \chi_1\mathbf{\Pi}^H\epsilon(\mathbf{x}) : \mathbf{\Pi}^H\epsilon(\mathbf{x})\rangle = \frac{1}{\theta_1}\left|\langle\chi_1\mathbf{\Pi}^H\epsilon(\mathbf{x})\rangle\right|^2 = \theta_1\left|\mathbf{\Pi}^H\bar{\epsilon}^1\right|^2, \tag{6.19}$$

where $\bar{\epsilon}^1$ is the constant field inside material one. On the other hand, since $\mu_1 = \mu_2$ one observes that (5.23) and (5.22) imply

$$\frac{1}{\theta_1}\left|\langle\chi_1\mathbf{\Pi}^H\epsilon(\mathbf{x})\rangle\right|^2 = \theta_1\left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}\right)^2\left|\mathbf{\Pi}^H\bar{\epsilon}\right|^2. \tag{6.20}$$

It easily follows from (6.19) and (6.20) that the hydrostatic component of the local strain attains the lower bound (3.11). Given $\mu_1 = \mu_2$ these arguments show that if the strain field is constant inside material one then its hydrostatic part attains the lower bound (3.11). Similar arguments show the optimality of the bound (3.12). The fact that the dilatational strain inside a rank-one laminate attains the two bounds (3.11) and (3.12), implies that it also attains the L^∞ bound (3.13). We suppose that $\kappa_1 = \kappa_2$, $d = 2$ and we denote the orthonormal system of eigenvectors for a prescribed 2×2 applied macroscopic strain by ψ^1, ψ^2. We show that the lower bounds presented in Section (3.5) are attained by the stress fields inside a rank-one laminate with layering direction $\mathbf{n} = \frac{1}{\sqrt{2}}(\psi^1 + \psi^2)$, see Fig. 2. Choosing $\kappa_1 = \kappa_2$ and $\mathbf{n} = \frac{1}{\sqrt{2}}(\psi^1 + \psi^2)$ in (6.15) gives

$$\lambda \odot \mathbf{n} = -\frac{\Delta\mu}{\langle\tilde{\mu}\rangle}\Pi^D\bar{\epsilon}. \tag{6.21}$$

It now follows from Eq. (6.13) that

$$\Pi^D\bar{\epsilon}^1 = \frac{\mu_2}{\langle\tilde{\mu}\rangle}\Pi^D\bar{\epsilon} \tag{6.22}$$

From this observation it is evident that the Von Mises equivalent strain field inside this rank-one laminate attains the bound (3.14).

6.3. The confocal ellipsoid (ellipse) assemblage and optimal lower bounds on local strain fields for subsets of applied macroscopic loads

In this section, it is shown that the lower bounds (3.7)–(3.10) are attained by the strain fields inside the confocal-ellipsoid and confocal-ellipse assemblages, see Fig. 3. Assuming that the uniform strain lies in $\mathcal{E}_1$ it follows that there is a confocal-ellipsoid (confocal-ellipse) assemblage with core of material one and coating of material two associated with $\bar{\epsilon}$ such that the local strain inside the core material is constant and hydrostatic. Since the strain field in material one is constant, then it follows from earlier arguments that

Fig. 2. A rank-one layered material.

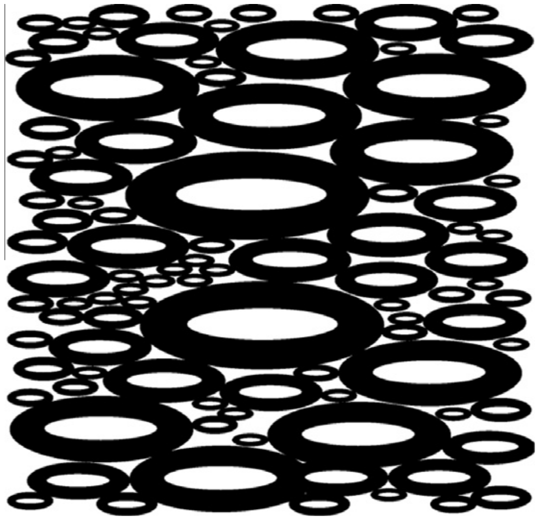

Fig. 3. Confocal-ellipse assemblage.

$$\langle\chi_1\Pi^H\epsilon(\mathbf{x}) : \epsilon(\mathbf{x})\rangle = \theta_1\left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}\right)^2 \left|\Pi^H\bar{\epsilon}\right|^2. \tag{6.23}$$

On the other hand, since the strain field in material one is hydrostatic one sees that

$$\langle\chi_1\Pi^D\epsilon(\mathbf{x}) : \epsilon(\mathbf{x})\rangle = 0 \tag{6.24}$$

and it is also evident that the lower bound (3.9) is attained. From Eqs. (6.23) and (6.24), and the fact that $\epsilon(\mathbf{x}) = \Pi^H\epsilon(\mathbf{x}) + \Pi^D\epsilon(\mathbf{x})$ one obtains

$$\langle\chi_1\epsilon(\mathbf{x}) : \epsilon(\mathbf{x})\rangle = \theta_1\left(\frac{\kappa_2 + 2\frac{d-1}{d}\mu}{\theta_1\kappa_2 + \theta_2\kappa_1 + 2\frac{d-1}{d}\mu}\right)^2 \left|\Pi^H\bar{\epsilon}\right|^2, \tag{6.25}$$

from which optimality of the bound (3.7) follows. Identical arguments show that the strain field inside material two of a confocal-ellipsoid (confocal-ellipse) assemblage with core of material two and coating of material one attains the bounds (3.8) and (3.10).

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